Journal of Mathematical Sciences: Advances and Applications
Volume 10, Number 1/2, 2011, Pages 53-74

EFFICIENT SUBSET OF PORTFOLIO UNDER DEGENERATE MEAN-VARIANCE MODEL

CHUN-FU JIANG and HONG-YI PENG

College of Mathematics and Computational Science
Shenzhen University
Shenzhen 518060
P. R. China
e-mail: jiangcf@szu.edu.cn

College of Sciences
South China Agricultural University
Guangzhou 510642
P. R. China

Abstract

The purpose of this paper is to discuss the problem how to check redundant assets for a mean-variance optimizing investor when the covariance matrix is the case of degeneracy. We propose a new concept of efficient subset of portfolio. We obtain some sufficient and necessary conditions for determining efficient subset. These conditions can be employed to decide whether new assets should be added to original portfolios. Moreover, the equivalent conditions analogous with k-funds separation theorem are derived. The extensions of these results to mean-variance spanning with singular covariance matrix are also considered.

2010 Mathematics Subject Classification: 91G10, 91G70.

Keywords and phrases: portfolio, efficient subset, singular covariance matrix, mean-variance spanning.

The research is supported by the Natural Science Foundation of Guangdong Province (No. 2008276) and The National Natural Science Foundation of China (No. 71101095).

Received August 4, 2011

1. Introduction

The mean-variance model for the portfolio selection problem pioneered by Markowitz is the most used and well-known tool for economic allocation of capital. In the previous research, the covariance matrix of asset returns are usually assumed to be non-singular (see, for example, Li et al. [14], Korki and Turtle [13], Bick [1], Zhang et al. [26], and Fang [6]). However, with the increase of asset classes and the rapid development of the derivatives markets for warrants, options, and futures, the degenerate portfolio selection can arise from the multicollinear or correlation between risky assets. Markowitz et al. [16] also pointed that one can not expect the covariance matrix to be positive definite in some important applications such as the case without short sale, where slack variables with zero variance are introduced. Nakasato and Furukawa [17] show that the degenerate cases can be observed when the covariance matrix is estimated from the returns series of a few number of periods, and they also find that a zero-variance portfolio emerges in the efficient frontier. Nevertheless, there is surprisingly little literature on such a general situation because conventional treatment methods are no longer applicable.

Buser [2] was the first to study the problem of portfolio with singular matrix, and showed that mutual-fund separation theorem still holds through constructing technically two new funds. Ryan and Lefoll [22] pointed out the errors existed in the demonstration of Buser [2]. VöRöS [25] considered the problem with special structure of covariance matrix. Korki and Turtle [12] developed the limiting investment opportunity set due to small risk assets, and demonstrated that the limiting result is similar to the investment opportunity set that arises when two assets are perfectly correlated. In addition, using simple tensor algebra, Los [15] investigated the multi-currency investment strategies with singular strategy risk matrix.

The motivation for this paper comes from the conjecture in Szegö [23] that, there is either arbitrage portfolio or efficient subset of portfolio when rank of covariance matrix is less than $n-1$, where n denotes number of risky assets. We also note the fact that the number of assets constituting the practical portfolio is often very small, comparing the number of candidate assets. For example, Nakasato and Furukawa [17] showed that the number of active securities is closely to the rank of covariance matrix. Therefore, an interesting and natural issue is whether there exist redundant assets in the pool of assets or, equivalently, whether there exist an asset subset forming the same efficient frontier as the complete asset set. However, the definition of efficient subset of portfolio cannot be found in printed literatures so far.

In this paper, we will present an explicit definition of the efficient subset of portfolio and study how to determine the efficient subset. Considering econometric testing and applications of efficient subset, we will also explore the equivalent conditions of efficient subset formulated by random returns of assets, which can be critical to empirical investigation on efficient subset and econometric testing for mean-variance spanning.

The outline of the paper is organized as follows. In Section 2, we will present some definitions and notations on the problems of portfolio selection. In Section 3, we show some sufficient and necessary conditions for determining the efficient subset of portfolio and the relation to mean-variance spanning.

2. Portfolio Selection Problem Under Degenerate Mean-Variance Model

Consider a portfolio selection problem with n assets (risky or riskless). The random return of the j-th asset is denoted by r_j. Let $\omega = (\omega_1, \omega_2, \cdots, \omega_n)'$ be investment weight vector, where ω_j be the fraction of wealth invested in asset j. Let $\boldsymbol{r} = (r_1, r_2, \cdots, r_n)'$ denote the n-vector of returns on the n assets, $\mu = (\mu_1, \mu_2, \cdots, \mu_n)'$ denote the n-vector of expected returns on the n assets, and $V = (\sigma_{ij})_{n\times n}$ denote the variance-covariance matrix, where prime indicates matrix transposition, and $\mu_i = \mathbb{E}(r_i)$, $\sigma_{ij} = \mathrm{Cov}(r_i, r_j)$, $i, j = 1, 2, \cdots, n$.

A portfolio is defined as a vector of the investment weight, and the return of portfolio is defined by $r_\omega = \omega' \boldsymbol{r}$. The expected return and the risk of a portfolio are, respectively, given by $\mu_\omega = \mathbb{E}(r_\omega) = \omega'\mu$ and $\sigma_\omega^2 = \mathrm{Var}(r_\omega) = \omega' V\omega$.

Without loss of generality, let $S_n = \{1, 2, \cdots, n\}$ be the set of all n assets, and $S_k = \{1, 2, \cdots, k\}$ be the subset of S_n. The set of portfolio based on S_n is defined as

$$W = \{\omega = (\omega_1, \omega_2, \cdots, \omega_n)' \in \mathbb{R}^n \mid \mathbf{1}'\omega = 1\},$$

and the set of portfolio based on subset S_k is defined as

$$W^k = \{\omega^k = (\omega_1, \omega_2, \cdots, \omega_k)' \in \mathbb{R}^k \mid \mathbf{1}_k' \omega^k = 1\},$$

where $\mathbf{1} = (1, 1, \cdots, 1)'$ is the vector of ones.

We assume the rank of covariance matrix V is arbitrary, that is, the covariance matrix can be singular or the case of so-called degenerate. If V is singular, in particular, then the portfolio ω satisfying $\omega' V = 0$ is called the *risk-free portfolio*. The set of risk-free portfolio on S_n is denoted by

$$W_f = \{\omega = (\omega_1, \omega_2, \cdots, \omega_n)' \in \mathbb{R}^n \mid \mathbf{1}'\omega = 1, V\omega = 0\}.$$

A portfolio selection problem in the mean-variance context can be written as

$$\begin{cases} \min\ \sigma_\omega^2 = \omega' V\omega, \\ \text{s.t.}\quad \omega'\mathbf{1} = 1, \\ \qquad\ \ \omega'\mu = r_p, \end{cases} \tag{2.1}$$

where r_p is the given expected return for a risk-averse investor.

The optimization problem (2.1) is obviously a quadratic programming with linear equality constraints. We can solve the problem by using Lagrange multiplier procedure. The Lagrangian function of the optimization problem is

$$L(\omega, \lambda_1, \lambda_2) = \frac{1}{2}\omega' V\omega - \lambda_1(\omega'\mathbf{1} - 1) - \lambda_2(\omega'\mu - r_p).$$

From the first-order conditions of the Lagrangian function, it follows that a portfolio ω is mean-variance efficient, if there exist scalars λ_1 and λ_2 such that

$$\begin{cases} V\omega - \lambda_1\mu - \lambda_2\mathbf{1} = 0, \\ \omega'\mathbf{1} - 1 = 0, \\ \omega'\mu - r_p = 0. \end{cases} \tag{2.2}$$

Since V is singular or non-singular, the Moore-Penrose inverse of matrix will be employed as an important tool of analysis.

Definition 2.1. An $n \times m$ matrix X is the Moore-Penrose inverse of a real $m \times n$ matrix A, if

$$AXA = A, \quad XAX = X, \quad (AX)' = AX, \quad (XA)' = XA.$$

We denote the Moore-Penrose inverse of A as A^+.

The vector space generated by the columns of $m \times n$ matrix A is denoted as

$$\mathscr{M}(A) = \{ y \,|\, y = Ax \quad \text{for some } x \in \mathbb{R}^n \}.$$

Lemma 2.1 (Dunne and Stone [5]). *Let A be an $n \times n$ matrix and c be an $n \times 1$ vector. Then*

(1) *if $c \in \mathscr{M}(A)$, then*

$$(A \pm cc')^+ = A^+ - \frac{A^+cc'A^+}{c'A^+c \pm 1};$$

(2) *if $c \notin \mathscr{M}(A)$, then*

$$(A + cc')^+ = A^+ + \frac{(1 + c'A^+c)P^\perp cc'P^\perp}{(c'P^\perp c)^2} - \frac{A^+cc'P^\perp + P^\perp cc'A^+}{c'P^\perp c},$$

where $P^\perp = I - AA^+$.

When the covariance matrix is singular, using the Moore-Penrose inverse, Jiang and Dai [10] obtain the analytic solutions of efficient portfolio and efficient frontier as the following lemma:

Lemma 2.2. *For the portfolio selection problem* (2.1), *if the inequality* $\mu \neq c\mathbf{1}$ *holds for any* $c \in \mathbb{R}$, *then we have*

(1) *If* $\mathbf{1} \in \mathscr{M}(V)$, $\mu \in \mathscr{M}(V)$, *then the frontier portfolio is*

$$\omega = \frac{Ar_p - B}{\Delta} V^+\mu + \frac{C - Br_p}{\Delta} V^+\mathbf{1} + P^{\perp}\xi,$$

where $A = \mathbf{1}'V^+\mathbf{1}$, $B = \mathbf{1}'V^+\mu$, $C = \mu'V^+\mu$, $\Delta = AC - B^2$, $P^{\perp} = I - VV^+$, *and* ξ *is any vector in* $\mathbb{R}^n$.

(2) *If* $\mathbf{1} \notin \mathscr{M}(V)$, *and* $\eta = \mu - \mu_{\pi}\mathbf{1} \in \mathscr{M}(V)$ *for any* $\pi \in W_f$, *then the frontier portfolio is*

$$\omega = \vartheta + (1 - \vartheta'\mathbf{1})\pi,$$

where ϑ *is the investment proportion of risky assets*, $1 - \mathbf{1}'\vartheta$ *is the investment proportion of risk-free portfolio* π, *and*

$$\vartheta = \frac{r_p - \mu_{\pi}}{\eta' V^+\eta} V^+\eta + P^{\perp}\xi, \quad \pi = \frac{P^{\perp}\mathbf{1}}{\mathbf{1}'P^{\perp}\mathbf{1}} + \left(P^{\perp} - \frac{P^{\perp}\mathbf{1}\mathbf{1}'P^{\perp}}{\mathbf{1}'P^{\perp}\mathbf{1}}\right)\zeta,$$

where $\xi, \zeta \in \mathbb{R}^n$ *are arbitrary. In particular, the return of risk-free portfolio is* $\mu_{\pi} = \dfrac{\mu'P^{\perp}\mathbf{1}}{\mathbf{1}'P^{\perp}\mathbf{1}}$.

There are other cases, that is, $\mathscr{M}(\mu\ \mathbf{1}) \cap \mathscr{M}(V) = \{0\}$ and $\mathbf{1} \in \mathscr{M}(V)$, $\mu \notin \mathscr{M}(V)$. For these cases, we have shown that the efficient frontiers are formulated by $\sigma_{\omega}^2 = 0$ and $\sigma_{\omega}^2 = 1/A$, respectively. However, it is impossible to occur under the *assumption of non-arbitrage* of the risk-free portfolio with the same returns as the risk-free asset.

3. Determination of Efficient Subset of Portfolio

3.1. Efficient subset of portfolio

In this section, we investigate the problem of efficient subset, that is, whether there is a subset of assets such that its mean-variance frontier is identical to the mean-variance frontier of the complete set S_n.

Let $S_k = \{1, 2, \cdots, k\}$ denote the benchmark assets class, and $S_n \setminus S_k = \{k+1, k+2, \cdots, n\}$ denote the additional assets class. Let

$$\boldsymbol{r}^k = (r_1, r_2, \cdots, r_k)', \quad \boldsymbol{r}^{n-k} = (r_{k+1}, r_{k+2}, \cdots, r_n)',$$

and $\boldsymbol{\mu}^k = \mathbb{E}(\boldsymbol{r}^k)$, $\boldsymbol{\mu}^{n-k} = \mathbb{E}(\boldsymbol{r}^{n-k})$. The covariance matrix V can be partitioned similarly as

$$V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix},$$

where $V_{11} = \mathrm{Var}(\boldsymbol{r}^k)$, $V_{21} = \mathrm{Cov}(\boldsymbol{r}^k, \boldsymbol{r}^{n-k})$, and $V_{22} = \mathrm{Var}(\boldsymbol{r}^{n-k})$.

Definition 3.1. Let W and W^k be the set of portfolio based on S_n and S_k, respectively. If for any portfolio $\omega \in W$, there is $\omega^k \in W^k$ such that

$$\mathbb{E}(\boldsymbol{r}' \omega) \leq \mathbb{E}((\boldsymbol{r}^k)' \omega^k), \quad \mathrm{Var}(\boldsymbol{r}' \omega) \geq \mathrm{Var}((\boldsymbol{r}^k)' \omega^k),$$

then we call S_k is the efficient subset of S_n.

Obviously, if S_k is the efficient subset of S_n, then the Definition 3.1 indicates the efficient frontier based on S_k is exactly the same as the efficient frontier based on S_n.

Theorem 3.1. *Let $S_n = \{1, 2, \cdots, n\}$ be the complete set of assets and $S_k = \{1, 2, \cdots, k\}$ be a subset of S_n. Then S_k is an efficient subset of S_n, if and only if*

$$\mathrm{Rank}(V_{11} \quad \mu^k \quad \mathbf{1}_k) = \mathrm{Rank}\begin{pmatrix} V_{11} & \mu^k & \mathbf{1}_k \\ V_{21} & \mu^{n-k} & \mathbf{1}_{n-k} \end{pmatrix}. \tag{3.1}$$

Proof. To prove the necessity, suppose that S_k is the efficient subset of S_n. Then for any efficient portfolio $\omega^k = (\omega_1, \omega_2, \cdots, \omega_k) \in W^k$ based on subset S_k, the expanded portfolio $\omega = (\omega_1, \omega_2, \cdots, \omega_k, 0, \cdots, 0)'$ is an efficient portfolio on S_n. Since ω^k is mean-variance efficient, ω^k obviously satisfies the equations similar to (2.2), that is,

$$\begin{cases} V_{11}\omega^k - \lambda_{1k}\mu^k - \lambda_{2k}\mathbf{1}_k = 0, \\ (\omega^k)'\mathbf{1}_k - 1 = 0, \\ (\omega^k)'\mu^k - r_p = 0, \end{cases} \tag{3.2}$$

where λ_{1k} and λ_{2k} are Lagrangian multipliers associated with the optimization portfolio problem on assets subset S_k.

On the other hand, the expanded portfolio

$$\omega = (\omega_1, \omega_2, \cdots, \omega_k, 0, \cdots, 0)',$$

also satisfies the equation (2.2). Substituting ω into (2.2), we have

$$\begin{cases} V_{11}\omega^k - \lambda_1\mu^k - \lambda_2\mathbf{1}_k = 0, \\ V_{21}\omega^k - \lambda_1\mu^{n-k} - \lambda_2\mathbf{1}_{n-k} = 0, \\ (\omega^k)'\mathbf{1}_k - 1 = 0, \\ (\omega^k)'\mu^k - r_p = 0. \end{cases} \tag{3.3}$$

Note that $\mu^k \neq c\mathbf{1}_k$ for any $c \in \mathbb{R}$, that is, μ^k and $\mathbf{1}$ are linearly independent. From the first equations of (3.2) and (3.3), it is easy to see that $\lambda_{1k} = \lambda_1, \lambda_{2k} = \lambda_2$.

By this, we show that ω^k is the solution of Equation (3.2), if and only if ω is the solution of Equation (3.3), which then implies (3.1).

To prove the sufficiency, suppose that (3.1) holds. For any efficient portfolio $\omega^k \in W^k$, following in the analysis of Section 1, there exist scalars λ_{1k} and λ_{2k} such that the Equation (3.2) holds. Combining Equations (3.2) and (3.3), the sufficiency is straightforward. The proof is finished.

3.2. Link to mean-variance spanning

Suppose an investor choose his portfolio from a set of k benchmark assets denoted by $S_k = \{1, 2, \cdots, k\}$. The issue of mean-variance spanning introduced by Huberman and Kandel [9] is whether new asset classes should be added to the benchmark assets. In general, if the mean-variance frontier of the benchmark assets coincides with the frontier of the benchmark plus the new asset classes, this is known as *mean-variance spanning*. The issue is also typically addressed by checking whether or not the additional asset classes improve the efficient frontier for the investor. The mean-variance spanning has recently received considerable attention in the literature. De Roon and Nijman [4] provided a comprehensive survey of the question of mean-variance spanning and how it relates to other fundamental concepts like stochastic discount factors. Cheung et al. [3] derived an analytical solution to the question that whether the investor should invest in the extra asset classes since spanning implies equal performance of the benchmark portfolio and the expanded portfolio.

If the benchmark assets can span the efficient frontier of all assets, it follows that the set of benchmark assets is an efficient subset of the set of all assets. Suppose that the covariance matrix $V > 0$, the necessary and sufficient conditions of mean-variance spanning were obtained by Huberman and Kandel [9] as the following:

$$V_{21}V_{11}^{-1}\mu^k = \mu^{n-k}, \quad V_{21}V_{11}^{-1}\mathbf{1}_k = \mathbf{1}_{n-k}, \tag{3.4}$$

and were cited in later literatures, such as De Roon and Nijman [4], Cheung et al. [3], Kan and Zhou [11], and Glabadanidis [7]. However, in more general cases, the covariance matrix can be singular. If we consider the problem of portfolio with the risk-free portfolio or risk-free asset, for example, the covariance matrix of all assets is not invertible. In these cases, the conditions of mean-variance spanning or efficient subset have not be addressed in current literatures.

Under mean-variance model, it is obvious that the efficient subset of portfolio is consistent with mean-variance spanning. According to Lemma 2.2, because of the term of $P^{\perp}\xi$, the efficient portfolio is not unique when covariance matrix is singular. It is easy to see that the term of $P^{\perp}\xi$ does not change the portfolio frontier when the real vector ξ is taken in $\mathbb{R}^n$ arbitrarily. Let $S_n = \{1, 2, \cdots, n\}$ be the complete set of assets, $S_k = \{1, 2, \cdots, k\}$ be a subset of S_n, and W^e be the set of frontier portfolio on S_n. Let W_f^k be the set of risk-free portfolio on S_k. Then, we have the following equivalent definition of efficient subset of portfolio:

Definition 3.2. For any frontier portfolio $p \in W^e$, if there exists a frontier portfolio $\omega \in W^e$ such that $\mu_\omega = \mu_p$ and $\omega^{n-k} = 0$, then we call S_k the efficient subset of S_n, where $\omega = \left((\omega^k)', (\omega^{n-k})'\right)'$.

Theorem 3.2. *Let $S_k = \{1, 2, \cdots, k\}$ be an asset subset of $S_n = \{1, 2, \cdots, n\}$. Then, we have*

(1) *If $\mathbf{1} \in \mathscr{M}(V), \mu \in \mathscr{M}(V)$, then S_k is an efficient subset of S_n, if and only if*

$$V_{21}V_{11}^{+}\mu^k = \mu^{n-k}, \quad V_{21}V_{11}^{+}\mathbf{1}_k = \mathbf{1}_{n-k}. \tag{3.5}$$

(2) *If $\mathbf{1} \notin \mathscr{M}(V), \eta = \mu - \mu_\pi \mathbf{1} \in \mathscr{M}(V)$ for any $\pi \in W_f$, then S_k is an efficient subset of S_n, if and only if W_f^k is non-empty and*

$$V_{21}V_{11}^{+}\eta^k = \eta^{n-k}, \tag{3.6}$$

where $\eta^k = \mu^k - \mu_\pi \mathbf{1}_k$, $\eta^{n-k} = \mu^{n-k} - \mu_\pi \mathbf{1}_{n-k}$.

Proof. (1) From Lemma 2.2, if $\mathbf{1} \in \mathscr{M}(V), \mu \in \mathscr{M}(V)$, thus the efficient portfolio is given by

$$\omega = \lambda_1 V^{+}\mu + \lambda_2 V^{+}\mathbf{1} + P^{\perp}\xi. \tag{3.7}$$

By [Groß [8], Theorem 1], we have

$$V^{+} = \begin{bmatrix} V_{11}^{+} + V_{11}^{+}V_{12}V_{22.1}^{\sim}V_{21}V_{11}^{+} & -V_{11}^{+}V_{12}V_{22.1}^{\sim} \\ -V_{22.1}^{\sim}V_{21}V_{11}^{+} & V_{22.1}^{\sim} \end{bmatrix}$$

$$+ \begin{bmatrix} -V_{11}^{+}(V_{12}Z + Z'V_{21})V_{11}^{+} & V_{11}^{+}Z' \\ ZV_{11}^{+} & 0 \end{bmatrix},$$

where

$$V_{22.1}^{\sim} = [Z \;\; I]\begin{bmatrix} V_{11}^{+} + V_{11}^{+}V_{12}V_{22.1}^{+}V_{21}V_{11}^{+} & -V_{11}^{+}V_{12}V_{22.1}^{+} \\ -V_{22.1}^{+}V_{21}V_{11}^{+} & V_{22.1}^{+} \end{bmatrix}\begin{bmatrix} Z' \\ I \end{bmatrix},$$

$$Z = (I - V_{22.1}V_{22.1}^{+})V_{21}V_{11}^{+}(I + V_{11}^{+}V_{12}(I - V_{22.1}V_{22.1}^{+})V_{21}V_{11}^{+})^{-1},$$

$$V_{22.1} = V_{22} - V_{21}V_{11}^{+}V_{12}.$$

Let $P = (I - V_{22.1}V_{22.1}^{+})V_{21}V_{11}^{+}$. Then, we notice that the following equalities:

$$(I + P'P)^{-1} = I - P'(I + PP')^{-1}P, \quad P(I + P'P)^{-1} = (I + PP')^{-1}P, \tag{3.8}$$

hold. From this and some computation, we obtain

$$P^{\perp} = I - VV^{+} = \begin{bmatrix} I - M + I - V_{11}V_{11}^{+} & -MP \\ -PM & I - V_{22.1}V_{22.1}^{+} - PMP' \end{bmatrix}, \tag{3.9}$$

where $M = (I + P'P)^{-1}$.

Now, we partition the efficient portfolio

$$\omega = ((\omega^{k})', (\omega^{n-k})')'$$

given in (3.6). From (3.8) and the Moore-Penrose inverse V^{+}, we have

$$\omega^{n-k} = V_{22.1}^{\sim}(\lambda_1\mu_{2.1} + \lambda_2\mathbf{1}_{2.1}) + ZV_{11}^{+}\left(\lambda_1\mu^{k} + \lambda_2\mathbf{1}_{k}\right) + Q\xi, \tag{3.10}$$

where

$\mu_{2.1} = \mu^{n-k} - V_{21}V_{11}^{+}\mu^{k}$, $\mathbf{1}_{2.1} = \mathbf{1}_{n-k} - V_{21}V_{11}^{+}\mathbf{1}_{k}$, and $Q = \big[-PM \,\vdots\, I$
$- V_{22.1}V_{22.1}^{+} - PMP' \big]$.

If S_k is an efficient subset of S_n, there is some real vector $\xi \in \mathbb{R}^n$ such that $\omega^{n-k} = 0$ for any target return r_p. By the definition of $V_{\widetilde{22.1}}$, Z and Q, we know

$$\mathscr{M}(V_{\widetilde{22.1}}) \subset \mathscr{M}(V_{22.1}), \quad \mathscr{M}(Z) \subset \mathscr{M}(Q) \subset \mathscr{M}(V_{22.1})^{\perp}, \tag{3.11}$$

where $\mathscr{M}(V_{22.1})^{\perp}$ is the orthocomplement space of $\mathscr{M}(V_{22.1})$.

From (3.10) and (3.11), the equality $\omega^{n-k} = 0$ holding for any r_p implies $\mu_{2.1} = 0$ and $\mathbf{1}_{2.1} = 0$, or equivalently,

$$V_{21}V_{11}^{+}\mu^{k} = \mu^{n-k}, \quad V_{21}V_{11}^{+}\mathbf{1}_{k} = \mathbf{1}_{n-k}.$$

Thus the proof of necessity is finished. Sufficiency is clear from (3.10) and (3.11).

(2) In this case, where $\mathbf{1} \notin \mathscr{M}(V)$, $\eta \in \mathscr{M}(V)$, it follows from Lemma 2.2 that the efficient portfolio is

$$\omega = \vartheta + (1 - \vartheta'\mathbf{1})\pi,$$

where ϑ and π are the same as Lemma 2.2.

Let us partition the frontier portfolio

$$\vartheta = \left((\vartheta^{k})', (\vartheta^{n-k})'\right)',$$

and the risk-free portfolio

$$\pi = \left((\pi^{k})', (\pi^{n-k})'\right)'.$$

Similar to the proof of (1), there is real vector $\xi_1 \in \mathbb{R}^n$ such that $\vartheta^{n-k} = 0$.

On the other hand, notice that $\left((\pi^{k})', 0, \cdots, 0\right)' \in W_f$ for any risk-free portfolio $\pi^{k} \in W_f^{k}$, and the following fact

$$W_f = \left\{\pi \,\middle|\, \pi = \frac{P^{\perp}\mathbf{1}}{\mathbf{1}'P^{\perp}\mathbf{1}} + \left(P^{\perp} - \frac{P^{\perp}\mathbf{1}\mathbf{1}'P^{\perp}}{\mathbf{1}'P^{\perp}\mathbf{1}}\right)\xi_2,\ \xi_2 \in \mathbb{R}^n\right\}$$

from Jiang and Dai [10], where W_f^k denotes the set of risk-free portfolio in S_k as

$$W_f^k = \{\omega^k = (\omega_1, \omega_2, \cdots, \omega_k)' | \mathbf{1}_k' \omega^k = 1,\ V_{11}\omega^k = 0\}.$$

Hence, if W_f^k is non-empty, there is some $\xi_2 \in \mathbb{R}^n$ such that $\pi^{n-k} = 0.$ These facts imply the sufficiency is established, and the proof of necessity is easy.

3.3. Link to *k*-funds separation theorem

The presentation of conditions for the efficient subset in Theorems 3.1~3.2 may be implicit. Now, we will establish the necessary and sufficient conditions for the efficient subset of portfolio by random returns of all assets in S_n. The results can be considered as an analogy to k-funds separation theorem.

Theorem 3.3. *Let* $S_k = \{1, 2, \cdots, k\}$ *be a subset of the complete set of assets* $S_n = \{1, 2, \cdots, n\}$. *Then* S_k *is the efficient subset of* S_n, *if and only if for every* $i \in S_n$, *there are* $\beta_{i1}, \beta_{i2}, \cdots, \beta_{ik}$ *such that*

$$r_i = \sum_{j=1}^{k} \beta_{ij} r_j + \varepsilon_i, \quad \sum_{j=1}^{k} \beta_{ij} = 1, \tag{3.12}$$

where ε_i, *a disturbance, satisfies* $\mathbb{E}(\varepsilon_i) = 0,\ \mathrm{Cov}(\varepsilon_i, r_j) = 0$ *for all* $i \in S_n \setminus S_k$ *and* $j \in S_k$.

Proof. Suppose that there exist $\beta_{i1}, \beta_{i2}, \cdots, \beta_{ik} \in \mathbb{R}$ such that (3.12) holds for all $i \in S_n \setminus S_k$. Let

$$\beta_i = (\beta_{i1}, \beta_{i2}, \ldots, \beta_{ik})', \quad \boldsymbol{\beta} = (\beta_{k+1}, \beta_{k+2}, \ldots, \beta_n)', \quad \boldsymbol{\varepsilon} = (\varepsilon_{k+1}, \varepsilon_{k+2}, \ldots, \varepsilon_n)'. \tag{3.13}$$

Then, we have $\boldsymbol{r}^{n-k} = \boldsymbol{\beta}\boldsymbol{r}^k + \boldsymbol{\varepsilon}$, where $E(\boldsymbol{\varepsilon}) = 0$, $\mathrm{Cov}(\boldsymbol{\varepsilon}, \boldsymbol{r}^k) = 0$, and

$$\boldsymbol{\beta}\mu^k = \mu^{n-k}, \quad \boldsymbol{\beta}V_{11} = V_{21}, \quad \boldsymbol{\beta}\mathbf{1}_k = \mathbf{1}_{n-k}.$$

It is shown from Theorem 3.1 that S_k is the efficient subset of S_n.

To prove the converse, assume that S_k is efficient subset of S_n. Then, there exists $\boldsymbol{\beta}$ such that

$$\boldsymbol{\beta}\begin{bmatrix} V_{11} & \mathbf{1}_k & \mu^k \end{bmatrix} = \begin{bmatrix} V_{21} & \mathbf{1}_{n-k} & \mu^{n-k} \end{bmatrix}.$$

Let $\boldsymbol{\varepsilon} = \boldsymbol{\beta}\boldsymbol{r}^k - \boldsymbol{r}^{n-k}$. We have $E(\boldsymbol{\varepsilon}) = 0$, $\mathrm{Cov}(\boldsymbol{\varepsilon}, \boldsymbol{r}^k) = \boldsymbol{\beta}V_{11} - V_{21} = 0$. Rewrite the elements of matrix $\boldsymbol{\beta}$ and random vector ε as (3.14). Then, there exist $\beta_{i1}, \beta_{i2}, \cdots, \beta_{ik} \in \mathbb{R}$ such that (3.12) holds. The theorem follows.

Let us consider a special case, where S_k contains the risk-free asset r_f. Let the 0-th asset be risk-free asset. From Theorem 3.3, then we obtain immediately the following corollary:

Corollary 3.1. *Let* $S'_k = \{0, 1, \cdots, k\}$ *be a subset of the complete set of assets* $S'_n = \{0, 1, \cdots, n\}$. *Then* S'_k *is the efficient subset of* S'_n, *if and only if for every* $i \in S'_n$, *there are* $\beta_{i1}, \beta_{i2}, \cdots, \beta_{ik} \in \mathbb{R}$ *such that*

$$r_i = \left(1 - \sum_{j=1}^{k} \beta_{ij}\right)\mu_{\pi(k)} + \sum_{j=1}^{k} \beta_{ij} r_j + \varepsilon_i, \tag{3.14}$$

or equivalently, $\eta_i = \sum_{j=1}^{k} \beta_{ij}\eta_j + \varepsilon_i$, *where* $\eta_i = r_i - \mu_{\pi(k)}$ *is the excess return of the i-th asset, and* ε_i *is a disturbance satisfying* $\mathbb{E}(\varepsilon_i) = 0$, $\mathrm{Cov}(\varepsilon_i, R_j) = 0$ *for all* $i \in S'_n \setminus S'_k$ *and* $j \in S'_k$.

Remark 3.1. Let r_a and r_b be random returns of portfolio a and b, respectively. Rothschild and Stiglitz [20] define the following semi-order relation:

$$r_a \succeq r_b \Leftrightarrow r_b = r_a + \varepsilon, \quad \mathbb{E}[\varepsilon | r_a] = 0,$$

where ε is a noise term. Obviously, $\succeq$ establishes the preference relation between r_a and r_b for risk aversion individual and $r_a \succeq r_b$ means that r_b is more risky than r_a because of the random term in r_a. Using the notations of Corollary 3.1, Rothschild and Stiglitz [20, 21] defined similarly the efficient subset under Rothschild-Stiglitz risk (RS risk for simple), and shows that S'_k is the efficient subset of S'_n, if and only if for every $i \in S'_n$, there exist coefficients $u_{i1}, u_{i2}, \cdots, u_{ik}$ such that

$$\eta_i = u_{i1}\eta_1 + u_{i2}\eta_2 + \cdots + u_{ik}\eta_k + \varepsilon_i,$$

and

$$\mathbb{E}[\varepsilon_i | \delta_1 r_1 + \delta_2 r_2 + \cdots + \delta_k r_k] = 0,$$

for any RS efficient portfolio $\omega = (\omega_1, \omega_2, \cdots, \omega_n)'$, where R_i is the same as Corollary 3.1, and

$$\delta_j = u_{1j}\omega_1 + u_{2j}\omega_2 + \cdots + u_{nj}\omega_n, \quad j = 1, 2, \cdots, k.$$

From Theorem 3.3 and Corollary 3.1, we know that the above conditions are sufficient for the mean-variance efficient subset, but not necessary unless the all assets in S'_k are RS efficient, which implies

$$\text{Cov}[\varepsilon_i, r_j] = 0, \quad \text{for any } i \in S_n \setminus S_k, \ j \in S_k.$$

Hence, RS efficient subset is a particular case of Theorem 3.3. If $k = 1$, in addition, then we have the interesting fact that CAPM formula can be derived by Corollary 3.1.

Theorem 3.4. *Let $S_k = \{1, 2, \cdots, k\}$ be a subset of $S_n = \{1, 2, \cdots, n\}$. Assume that there is not any scalar $c \in \mathbb{R}$ such that $\mu = c\mathbf{1}$, and for any $i \in S_n$, there are scalars $\beta_{i0}, \beta_{i1}, \beta_{i2}, \cdots, \beta_{ik} \in \mathbb{R}$ such that*

$$r_i = \beta_{i0} + \sum_{j=1}^{k} \beta_{ij} r_j. \tag{3.15}$$

Then S_k *is the efficient subset of* S_n, *if and only if one of the following conditions is satisfied*:

(1) $$\sum_{j=1}^{k} \beta_{ij} = 1, \text{ for any } i \in S_n;$$

and

(2) $$\sum_{j=1}^{k} \beta_{ij} \neq 1, \text{ for some } i \in S_k.$$

Proof. For any $i \in S_n$, there exist $\beta_{i0}, \beta_{i1}, \beta_{i2}, \cdots, \beta_{ik} \in \mathbb{R}$ such that (3.15) holds. Let us denote

$$\boldsymbol{\beta}_{(1)} = (\beta_1, \beta_2, \cdots, \beta_k)', \quad \boldsymbol{\beta}_{(2)} = (\beta_{k+1}, \beta_{k+2}, \cdots, \beta_n)', \quad \boldsymbol{\beta} = (\boldsymbol{\beta}'_{(1)}, \boldsymbol{\beta}'_{(2)})', \tag{3.16}$$

and

$$\beta_0^{(1)} = (\beta_{10}, \beta_{20}, \cdots, \beta_{k0})', \quad \beta_0^{(2)} = (\beta_{k+1,0}, \beta_{k+2,0}, \cdots, \beta_{n0})',$$

$$\beta_0 = \left(\left(\beta_0^{(1)}\right)', \left(\beta_0^{(2)}\right)' \right)'. \tag{3.17}$$

Thus, we have $\boldsymbol{r} = \beta_0 + \boldsymbol{\beta r}^k$, and

$$\mathrm{Var}(\boldsymbol{r}) = \mathrm{Cov}(\boldsymbol{\beta r}^k + d_0, \boldsymbol{r}) = \begin{bmatrix} \boldsymbol{\beta}_{(1)} V_{11} & \boldsymbol{\beta}_{(1)} V_{12} \\ \boldsymbol{\beta}_{(2)} V_{11} & \boldsymbol{\beta}_{(2)} V_{12} \end{bmatrix} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}, \tag{3.18}$$

$$\mathbb{E}(\boldsymbol{r}) = \begin{bmatrix} \boldsymbol{\beta}_{(1)} \\ \boldsymbol{\beta}_{(2)} \end{bmatrix} \mu^k + \begin{bmatrix} \beta_0^{(1)} \\ \beta_0^{(2)} \end{bmatrix} = \begin{bmatrix} \mu^k \\ \mu^{n-k} \end{bmatrix}. \tag{3.19}$$

Since $\beta_{i0} = r_i - [\beta_{i1} r_1 + \beta_{i2} r_2 + \cdots + \beta_{ik} r_k]$ for any $i \in S_n$, and noting that β_{i0} is a scalar, thus $\mathrm{Var}(\beta_{i0}) = 0$, these imply β_{i0} is equivalent to $1 - (\beta_{i1} + \beta_{i2} + \cdots + \beta_{ik})$ times investment of risk-free asset or portfolio. Using the assumption of non-arbitrage, we have

$$\beta_{i0} = [1 - (\beta_{i1} + \beta_{i2} + \cdots + \beta_{ik})] r_f,$$

for any $i \in S_n$, that is,

$$\beta_0^{(1)} = (\mathbf{1}_k - \boldsymbol{\beta}_{(1)}\mathbf{1}_k) r_f, \quad \beta_0^{(2)} = (\mathbf{1}_{n-k} - \boldsymbol{\beta}_{(2)}\mathbf{1}_k) r_f. \tag{3.20}$$

If S_n does not contain risk-free asset, replacing r_f by μ_π, the return of risk-free portfolio π, the equalities in (3.24) remain hold. If W_f is empty set, then we have $\beta_{i0} = 0$ from the analysis in the following

Substituting (3.20) into (3.19), we obtain

$$\boldsymbol{\beta}_{(1)}\eta^k = \eta^k, \quad \boldsymbol{\beta}_{(2)}\eta^k = \eta^{n-k}, \tag{3.21}$$

where $\eta^k = \mu^k - r_f\mathbf{1}_k$, $\eta^{n-k} = \mu^{n-k} - r_f\mathbf{1}_{n-k}$.

Next, to prove the theorem, we investigate the following four cases:

(i) If $\boldsymbol{\beta}_{(1)}\mathbf{1}_k = \mathbf{1}_k$, $\boldsymbol{\beta}_{(2)}\mathbf{1}_k = \mathbf{1}_{n-k}$, that is, $\beta_{i1} + \beta_{i2} + \cdots + \beta_{ik} = 1$ for any $i \in S_n$, then from (3.18) and (3.19), we obtain

$$\begin{bmatrix} V_{21} & \mathbf{1}_{n-k} & \mu^{n-k} \end{bmatrix} = \boldsymbol{\beta}_{(2)}\begin{bmatrix} V_{11} & \mathbf{1}_k & \mu^k \end{bmatrix}.$$

It is easy to see from Theorem 3.1 that S_k is the efficient subset of S_n.

(ii) If $\boldsymbol{\beta}_{(1)}\mathbf{1}_k \neq \mathbf{1}_k$, that is, there exists $i \in S_k$ such that $\beta_{i1} + \beta_{i2} + \cdots + \beta_{ik} \neq 1$, it thus follows that $\mathbf{1}_k \notin \mathscr{M}(V_{11})$. Otherwise, if $\mathbf{1}_k \in \mathscr{M}(V_{11})$, then there is $x \in \mathbb{R}^k$ such that $\mathbf{1}_k = V_{11}x$. Consequently, we have

$$\boldsymbol{\beta}_{(1)}\mathbf{1}_k = \boldsymbol{\beta}_{(1)}V_{11}x = V_{11}x = \mathbf{1}_k,$$

which is contradict with $\boldsymbol{\beta}_{(1)}\mathbf{1}_k \neq \mathbf{1}_k$.

On the other hand, from the assumption of non-arbitrage, we have $\eta^k \in \mathscr{M}(V_{11})$, which implies $\mu^k \neq c\mathbf{1}_k$ for any $c \in \mathbb{R}$. Otherwise, there is some scalar $c \in \mathbb{R}$ such that $\mu^k = c\mathbf{1}_k$, then we have $\eta^k = (c - r_f)\mathbf{1}_k$. From (3.21), we obtain

$$(c - r_f)\boldsymbol{\beta}_{(1)}\mathbf{1}_k = (c - r_f)\mathbf{1}_k, \quad (c - r_f)\boldsymbol{\beta}_{(2)}\mathbf{1}_k = (\mu^{n-k} - r_f\mathbf{1}_{n-k}). \tag{3.22}$$

Since $\boldsymbol{\beta}_{(1)}\mathbf{1}_k \neq \mathbf{1}_k$, from (3.22), we have $c = r_f$, thus $\mu = r_f\mathbf{1}$, which contradicts the assumption that there is not $c \in \mathbb{R}$ such that $\mu = c\mathbf{1}$.

Let $\Omega = V_{11} + \eta^k(\eta^k)'$. From above, we have $\mathbf{1}_k \notin \mathscr{M}(\Omega)$. Applying Lemma 2.1 yields

$$\mathbf{1}_k'\left(V_{11} + \mathbf{1}_k\mathbf{1}_k' + \eta^k(\eta^k)'\right)^+\mathbf{1}_k = \mathbf{1}_k'(\Omega + \mathbf{1}_k\mathbf{1}_k')^+\mathbf{1}_k = 1. \tag{3.23}$$

Note that $\eta^k \in \mathscr{M}(V_{11})$, $\mathscr{M}(V_{12}) \subset \mathscr{M}(V_{11})$. Therefore, we also have

$$\mathbf{1}_k'\left(V_{11} + \mathbf{1}_k\mathbf{1}_k' + \eta^k(\eta^k)'\right)^+ V_{11} = 0, \tag{3.24}$$

$$\mathbf{1}_k'\left(V_{11} + \mathbf{1}_k\mathbf{1}_k' + \eta^k(\eta^k)'\right)^+ V_{21} = 0, \tag{3.25}$$

$$\mathbf{1}_k'\left(V_{11} + \mathbf{1}_k\mathbf{1}_k' + \eta^k(\eta^k)'\right)^+ \eta^k = 0. \tag{3.26}$$

Let

$$A = (V_{21} + \mathbf{1}_{n-k}\mathbf{1}_k' + \eta^{n-k}\eta^k)(V_{11} + \mathbf{1}_k\mathbf{1}_k' + \eta^k\eta^k)^+. \tag{3.27}$$

Computing from (3.26) yields

$$\begin{aligned} A\eta^k &= \left(V_{21} + \mathbf{1}_k\mathbf{1}_k' + \eta^{n-k}(\eta^k)'\right)\left(V_{11} + \mathbf{1}_k\mathbf{1}_k' + \eta^k(\eta^k)'\right)^+\eta^k \\ &= V_{21}\left(V_{11} + \mathbf{1}_k\mathbf{1}_k' + \eta^k(\eta^k)'\right)^+\eta^k + \frac{(\eta^k)'\Delta^+\eta^k}{1 + (\eta^k)'\Delta^+\eta^k}\eta^{n-k} \\ &= \frac{\boldsymbol{\beta}_{(2)}(\Delta - \mathbf{1}_k\mathbf{1}_k')\Delta^+\eta^k}{1 + (\eta^k)'\Delta^+\eta^k} + \frac{(\eta^k)'\Delta^+\eta^k}{1 + (\eta^k)'\Delta^+\eta^k}\eta^{n-k} \\ &= \eta^{n-k}, \end{aligned} \tag{3.28}$$

where $\Delta = V_{11} + \mathbf{1}_k\mathbf{1}_k'$, and the second line is owing to the Moore-Penrose inverse

$$\left(V_{11} + \mathbf{1}_k\mathbf{1}_k' + \eta^k(\eta^k)'\right)^+ = \left(\Delta + \eta^k(\eta^k)'\right)^+ = \Delta^+ - \frac{\Delta^+\eta^k(\eta^k)'\Delta^+}{1 + (\eta^k)'\Delta^+\eta^k}.$$

Similarly, we can prove $AV_{11} = V_{21}$, $A\mathbf{1} = \mathbf{1}$ by using (3.23)-(3.25). Therefore, S_k is an efficient subset of S_n.

(iii) If $\boldsymbol{\beta}_{(1)}\mathbf{1}_k = \mathbf{1}_k$, $\boldsymbol{\beta}_{(2)}\mathbf{1}_k \neq \mathbf{1}_{n-k}$, then S_k is not efficient subset of S_n. In fact, noting that $\boldsymbol{\beta}\mathbf{1} = (\boldsymbol{\beta}_{(1)}', \boldsymbol{\beta}_{(2)}')'\mathbf{1} \neq \mathbf{1}$, by the similar proof of (ii), we have $\mathbf{1} \notin \mathscr{M}(V)$ and $\mathbf{1}_k \in \mathscr{M}(V_{11})$, which implies that the risk-free portfolio is available in S_k, but not in S_n. Consequently, S_k can not be an efficient subset of S_n. The proof is thus complete.

Remark 3.2. (i) Under the assumption of Theorem 3.4, it is clear that the following equality:

$$V_{22.1} = V_{22} - V_{21}V_{11}^+V_{12} = 0,$$

holds. Note that for any frontier portfolio ω, the portfolio risk can be written in the form

$$\sigma_\omega^2 = \omega' V\omega = \begin{bmatrix} \nu^k \\ \nu^{n-k} \end{bmatrix}' \begin{bmatrix} V_{11} & 0 \\ 0 & V_{22.1} \end{bmatrix} \begin{bmatrix} \nu^k \\ \nu^{n-k} \end{bmatrix}, \tag{3.29}$$

where

$$\begin{bmatrix} \nu^k \\ \nu^{n-k} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_k & V_{11}^+V_{12} \\ 0 & \mathbf{I}_{n-k} \end{bmatrix} \begin{bmatrix} \omega^k \\ \omega^{n-k} \end{bmatrix}.$$

From (3.29), we can further find

$$\sigma_\omega^2 = (\nu^k)'V_{11}\nu^k + (\nu^{n-k})'V_{22.1}\nu^{n-k} = (\nu^k)'V_{11}\nu^k. \tag{3.30}$$

Also note that the case (1) in Theorem 3.4 implies $V_{21}V_{11}^+\mathbf{1}_k = \mathbf{1}_{n-k}$. Then, we have

$$(\nu^k)'\mathbf{1}_k = (\omega^k)'\mathbf{1}_k + (\omega^{n-k})'V_{21}V_{11}^+\mathbf{1}_k = (\omega^k)'\mathbf{1}_k + (\omega^{n-k})'\mathbf{1}_{n-k} = 1. \tag{3.31}$$

Since ω is an arbitrary frontier portfolio, it follows from (3.30) and (3.31) that $\nu^k \in W^k$ is a frontier portfolio on S_k.

It also happens that the case (1) in Theorem 3.4 does not hold. In this situation, we use the following transformation:

$$V = \begin{bmatrix} \mathbf{I}_k & 0 \\ A & \mathbf{I}_{n-k} \end{bmatrix} \begin{bmatrix} V_{11} & 0 \\ 0 & V_{22.1} \end{bmatrix} \begin{bmatrix} \mathbf{I}_k & A' \\ 0 & \mathbf{I}_{n-k} \end{bmatrix}, \tag{3.32}$$

where A is defined in Equation (3.27). It is easy to verify

$$AV_{11} = V_{21}, \quad A\eta^k = \eta^{n-k}, \quad AV_{12} = V_{21}V_{11}^{+}V_{12}, \quad A\mathbf{1}_k = \mathbf{1}_{n-k},$$

similar to (3.28). Let $\nu^k = \omega^k + A'\omega^{n-k}$. Then, we can prove in the same way that $\nu^k \in W^k$ is a frontier portfolio on S_k. Moreover, these facts imply that removing S_{n-k} does not change the portfolio frontier on S_n. Consequently, S_k is the efficient subset of S_n.

(ii) In Theorem 3.3, we in fact assume that the random terms ε_i $(i \in S_n)$ with zero expectation and finite variance, and the joint covariance matrix can be derived as the following:

$$\mathrm{Var}(\varepsilon) = V_{22} - V_{21}V_{11}^{+}V_{12} = V_{22.1},$$

where $\varepsilon = (\varepsilon_{k+1}, \varepsilon_{k+2}, \cdots, \varepsilon_n)'$. According to the transformation of portfolio risk similar to (3.30), from Lemma 2.2 and (3.10)-(3.11), we can see that the sufficient and necessary conditions in Theorems 3.3 and 3.2 are equivalent.

Remark 3.3. According to the definition of efficient subset, it is clear that the efficient subset is not unique. In other words, there exist many efficient subsets of S_n in practice. Among them, there exists the efficient subset with minimal size. Suppose that the set of risk-free portfolio W_f is empty. It is shown from Theorem 3.2 that efficient subset S_k is the minimal one, if and only if $\mathrm{rank}(V) = \mathrm{rank}(V_{11}) = k$, or equivalently,

$$V_{22.1} = V_{22} - V_{21}V_{11}^{-1}V_{12} = 0. \tag{3.33}$$

On the other hand, if there exists risk-free portfolio $\pi \in W_f$ such that $\eta = \mu - \mu_\pi \mathbf{1} \in \mathscr{M}(V)$, then S_k is the minimal one, if and only if W_f^k is non- empty and $\text{rank}(V) = \text{rank}(V_{11}) = k - 1$. Replacing V^{-1} with generalized inverse V^+ in (3.33), the equality $V_{22.1} = 0$ remain holds.

References

[1] A. Bick, The mathematics of the portfolio frontier: A geometry-based approach, The Quarterly Review of Economics and Finance 44 (2004), 337-361.

[2] S. A. Buser, Mean-variance portfolio selection with either a singular or non-singular variance-covariance matrix, Journal of Financial and Quantitative Analysis 12 (1977), 347-361.

[3] C. S. Cheung, C. Kwan and D. Mountain, On the nature of mean-variance spanning, Finance Research Letter 6 (2009), 106-113.

[4] F. De Roon and T. Nijman, Testing for mean-variance spanning: A survey, Journal of Empirical Finance 8 (2001), 111-155.

[5] T. T. Dunne and M. Stone, Downdating the Moore-Penrose generalized inverse for cross-validation of centred least squares prediction, Journal of the Royal Statistical Society: Series B 55 (1993), 369-375.

[6] S. H. Fang, A mean-variance analysis of arbitrage portfolios, Physica A: Statistical Mechanics and its Applications 375 (2007), 625-632.

[7] P. Glabadanidis, Measuring the economic significance of mean-variance spanning, The Quarterly Review of Economics and Finance 49 (2009), 596-616.

[8] J. Groß, The Moore-Penrose inverse of a partitioned non-negative definite matrix, Linear Algebra and its Applications 321 (2000), 113-121.

[9] G. Huberman and S. Kandel, Mean variance spanning, Journal of Finance 42 (1987), 873-888.

[10] C. F. Jiang and Y. L. Dai, Analytic solutions of efficient frontier and efficient portfolio with singular covariance matrix, Journal of Systems Science and Mathematical Sciences 28 (2008), 1134-1147.

[11] R. Kan and G. Zhou, Tests of Mean-Variance Spanning, Working Paper, University of Toronto and Washington University in St. Louis, 2008.

[12] B. Korki and H. J. Turtle, A note on the analytics and geometry of limiting mean-variance investment opportunity sets, Review of Quantitative Finance and Accounting 9 (1994), 289-300.

[13] B. Korki and H. J. Turtle, A mean-variance analysis of self-financing portfolios, Management Science 48 (2002), 427-433.

[14] Z. F. Li, Z. X. Li, S. Y. Wang and X. T. Deng, Optimal portfolio selection of assets with transaction costs and no short sales, International Journal of Systems Science 32 (2001), 599-607.

[15] C. A. Los, Optimal multi-currency investment strategies with exact attribution in three Asian countries, Journal of Multinational Financial Management 8 (1998), 169-198.

[16] H. Markowitz, R. Lacey and J. Plymen, The general mean-variance portfolio selection problem (and discussion), Phil. Trans. R. Soc. Lond. A. (1994), 543-549.

[17] M. Nakasato and K. Furukawa, On the number of securities which constitute an efficient portfolio, Annals of Operations Research 45 (1993), 333-347.

[18] A. Perold, Large-scale portfolio optimization, Management Science 30 (1984), 1143-1160.

[19] S. Ross, Mutual fund separation in financial theory: The separating distributions, Journal of Economic Theory 17 (1978), 254-286.

[20] M. Rothschild and J. Stiglitz, Increasing risk I: A definition, Journal of Economic Theory 2 (1970), 225-243.

[21] M. Rothschild and J. Stiglitz, Increasing risk II: Its economics consequences, Journal of Economic Theory 3 (1971), 66-84.

[22] P. J. Ryan and J. Lefoll, A comment on mean-variance portfolio selection with either a singular or a non-singular variance-covariance matrix, Journal of Financial and Quantitative Analysis 16 (1981), 389-396.

[23] G. P. Szegö, Portfolio Theory: With Application to Bank Asset Management, Academic Press, New York, 1980.

[24] A. D. Ukhov, Expanding the frontier one asset at a time, Finance Research Letter 3 (2006), 194-206.

[25] J. VöRöS, The explicit derivation of the efficient portfolio frontier in the case of degeneracy and general singularity, European Journal of Operational Research 32 (1987), 302-310.

[26] S. M. Zhang, S. Y. Wang and X. T. Deng, Portfolio selection theory with different interest rates for borrowing and lending, Journal of Global Optimization 28 (2004), 67-95.

[27] W. G. Zhang, W. L. Xiao and W. L. Xu, A possibilisitic portfolio adjusting model with new added assets, Economic Modelling 27 (2010), 208-213.

■